Int. J. Heat Mass Transfer. Vol. 21, pp. 923–933
© Pergamon Press Ltd. 1978. Printed in Great Britain

0017-9310/78/0701-0923 $02.00/0

THERMAL BOUNDARY LAYER OF A MICROPOLAR FLUID ON A CIRCULAR CYLINDER

M. N. MATHUR
Mathematics Department, Indian Institute of Technology, Bombay, India

S. K. OJHA
Aeronautical Engineering Department, Indian Institute of Technology, Bombay, India

and

P. SUBHADRA RAMACHANDRAN
Mathematics Department, Indian Institute of Technology, Bombay, India

(*Received* 8 *February* 1977 *and in revised form* 26 *September* 1977)

Abstract—Steady thermal boundary-layer flow past a circular cylinder whose axis is placed normal to an oncoming free stream of an incompressible micropolar fluid has been studied. The solution of the energy equation inside the boundary-layer is obtained as a power series of the distance measured along the surface from the front stagnation point on the cylinder. The surface of the circular cylinder is maintained at a constant temperature and the temperature outside the boundary layer is also kept constant. The dimensionless temperature distribution and the heat-transfer coefficient have been presented graphically for various values of the material parameters. A comparison has been made with the corresponding results for Newtonian fluids. The temperature inside the boundary layer is more, and the heat-transfer coefficient is less, for micropolar fluids as compared with that for Newtonian fluids.

NOMENCLATURE

- A_n, B_n, C_n, D_n, E_n, functions of η appearing in equations (16e–g);
- C_p, specific heat at constant pressure;
- E, Eckert number;
- f_n, functions of η appearing in equations (15a,b);
- g, non-dimensional component of microrotation;
- g_0, non-dimensional component of microrotation inside the boundary layer;
- g_n, functions of η appearing in equation (15c);
- $h_n, k_n, j_n, l_n, m_n, n_n$, functions of η appearing in equations (16a–d);
- j, microinertia per unit mass;
- K, surface curvature;
- K_c, coefficient of heat conduction;
- k_v, vortex viscosity coefficient;
- L, radius of the circular cylinder;
- Nu, Nusselt number;
- $\bar{N}_1, N_1$, k_v/μ_v;
- $\bar{N}_2$, j/L^2;
- N_2, $\bar{N}_2/\varepsilon^2$;
- $\bar{N}_3$, $\gamma_v/\mu_v L^2$;
- N_3, $\bar{N}_3/\varepsilon^2$;
- p, non-dimensional pressure;
- p_0, non-dimensional pressure inside the boundary layer;
- Pr, Prandtl number;
- q, surface heat flux;
- R, Reynolds number;
- T, temperature;
- T_∞, temperature of the oncoming free stream;
- T_w, temperature of the wall;
- u, v, non-dimensional components of velocity along x and y directions respectively;
- u_0, v_0, non-dimensional components of velocity inside the boundary layer;
- U_0, non-dimensional inviscid flow velocity on the cylinder;
- U_∞, velocity of the oncoming free stream;
- x, non-dimensional distance measured along the surface from the front stagnation point;
- y, non-dimensional distance measured along the normal to the surface;
- Y, y/ε.

Greek symbols

- α^*, micropolar heat-conduction coefficient;
- α, $\alpha^* U_\infty/\mu_v L C_p$ (non-dimensional micropolar heat-conduction coefficient);
- γ_v, spin gradient viscosity coefficient;
- ε, $1/(R)^{1/2}$;
- η, $Y(a_1)^{1/2}$ ($a_1 = 2$ for a circular cylinder);
- θ, non-dimensional temperature;
- θ_0, non-dimensional temperature inside the boundary layer;
- μ_v, viscosity coefficient;
- ρ, mass density of the fluid;
- ϕ, angle measured in degrees from the front stagnation point.

1. INTRODUCTION

EXPERIMENTS due to Hoyt and Fabula [1], Vogel and Patterson [2], with fluids containing extremely small amount of polymeric additives indicate a reduction in skin friction near a rigid body when compared with the skin friction in the same fluids without additives. This phenomenon cannot be explained on the basis of classical continuum mechanics. In support of these above experiments Eringen [3] has proposed the theory of micropolar fluids which takes into account the inertial characteristics of the substructure particles which are also allowed to undergo rotation. This theory can be applied to explain the flow problems of colloidal fluids, liquid crystals, fluids with additives, animal blood etc.

The theory of thermo-micropolar fluids was introduced by Eringen [4] by formulating the full energy equation of a micropolar fluid. These non-linear equations of motion are too difficult to solve even in comparatively simpler situations. Peddieson and McNitt [5] obtained the boundary-layer equations of a micropolar fluid. The boundary-layer theory of micropolar fluids as proposed by Peddieson and McNitt did not consider the thermal effects.

In Section 2, we have derived the thermal boundary-layer equations for the steady flow of an incompressible micropolar fluid past a circular cylinder.

We assume Blasius type of power series [6] for velocity, microrotation and temperature inside the boundary layer. The substitution of these power series in the thermal boundary-layer equations gives rise to a system of ordinary differential equations. This system of equations along with the corresponding boundary conditions has been solved in Section 3 for various values of the material parameters entering into the problem. The numerical values chosen for these micropolar fluid parameters are some of those that have been chosen in [5]. It may be noted that thermal boundary layer of a micropolar fluid gives rise to an additional parameter α which is due to heat conduction in micropolar fluids. We assume constant temperature distribution outside the boundary layer and constant wall temperature.

2. FORMULATION OF THE PROBLEM

We choose an orthogonal curvilinear co-ordinate system (x, y) in which x is measured along the surface of the cylinder from the front stagnation point and y normal to the surface of the cylinder. The non-dimensional equations governing the steady flow of a micropolar fluid past a circular cylinder in this co-ordinate system are:

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}[(1+Ky)v]=0. \tag{1}$$

Momentum:

$$\frac{u}{(1+Ky)}\left(\frac{\partial u}{\partial x}+Kv\right)+v\frac{\partial u}{\partial y}=-\frac{1}{(1+Ky)}\frac{\partial p}{\partial x}+\frac{(1+\bar{N}_1)}{R}\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2K}{(1+Ky)^2}\frac{\partial v}{\partial x}\right.$$
$$\left.-\frac{y}{(1+Ky)^3}\frac{\partial u}{\partial x}\frac{\mathrm{d}K}{\mathrm{d}x}+\frac{K}{(1+Ky)}\frac{\partial u}{\partial y}+\frac{v}{(1+Ky)^3}\frac{\mathrm{d}K}{\mathrm{d}x}-\frac{K^2u}{(1+Ky)^2}\right]+\frac{\bar{N}_1}{R}\frac{\partial g}{\partial y}, \tag{2}$$

$$\frac{u}{(1+Ky)}\left(\frac{\partial v}{\partial x}-Ku\right)+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{(1+\bar{N}_1)}{R}\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-\frac{2K}{(1+Ky)^2}\frac{\partial u}{\partial x}\right.$$
$$\left.-\frac{K^2 v}{(1+Ky)^2}-\frac{y}{(1+Ky)^3}\frac{\partial v}{\partial x}\frac{\mathrm{d}K}{\mathrm{d}x}+\frac{K}{(1+Ky)}\frac{\partial v}{\partial y}-\frac{u}{(1+Ky)^3}\frac{\mathrm{d}K}{\mathrm{d}x}\right]-\frac{\bar{N}_1}{R}\frac{1}{(1+Ky)}\frac{\partial g}{\partial x}. \tag{3}$$

Moment of momentum:

$$\bar{N}_2\left[\frac{1}{(1+Ky)}u\frac{\partial g}{\partial x}+v\frac{\partial g}{\partial y}\right]=\frac{\bar{N}_3}{R}\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{K}{(1+Ky)}\frac{\partial g}{\partial y}-\frac{y}{(1+Ky)^3}\frac{\partial g}{\partial x}\frac{\mathrm{d}K}{\mathrm{d}x}\right]$$
$$+\frac{\bar{N}_1}{R}\left[\frac{1}{(1+Ky)}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}-\frac{Ku}{(1+Ky)}-2g\right]. \tag{4}$$

Energy:

$$\frac{u}{(1+Ky)}\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\frac{E}{R}\left(1+\frac{\bar{N}_1}{2}\right)\left\{2\left[\frac{1}{(1+Ky)}\left(\frac{\partial u}{\partial x}+Kv\right)\right]^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left[\frac{1}{(1+Ky)}\left(\frac{\partial v}{\partial x}-Ku\right)+\frac{\partial u}{\partial y}\right]^2\right\}$$
$$+\frac{2\bar{N}_1 E}{R}\left\{g-\frac{1}{2}\left[\frac{1}{(1+Ky)}\left(\frac{\partial v}{\partial x}-Ku\right)-\frac{\partial u}{\partial y}\right]\right\}^2+\frac{\bar{N}_3 E}{R}\left\{\left[\frac{1}{(1+Ky)}\frac{\partial g}{\partial x}\right]^2+\left[\frac{\partial g}{\partial y}\right]^2\right\}$$
$$+\frac{\alpha}{R(1+Ky)}\left[\frac{\partial\theta}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial\theta}{\partial y}\frac{\partial g}{\partial x}\right]+\frac{1}{PrR}\left[\frac{1}{(1+Ky)^2}\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}-\frac{y}{(1+Ky)^3}\frac{\mathrm{d}K}{\mathrm{d}x}\frac{\partial\theta}{\partial x}+\frac{K}{(1+Ky)}\frac{\partial\theta}{\partial y}\right]. \tag{5}$$

The boundary conditions on the surface of the cylinder are

$$u=v=g=0,\quad \theta=1 \tag{6}$$

and for outer boundary conditions we take the free stream values of velocity, microrotation and temperature.

In the above equations u and v are the components of velocity along x and y directions respectively and g is the component of microrotation whose direction of rotation is in the x–y plane. p is the pressure and θ is the temperature. We have non-dimensionalised the co-ordinates by the radius of the circular cylinder L, velocities by U_∞ (oncoming free stream velocity), microrotation by U_∞/L and the pressure by ρU_∞^2. The dimensionless temperature θ is defined as $\theta = (T-T_\infty)/(T_w-T_\infty)$. T_w and T_∞ denote respectively the constant temperatures of the wall and the free stream.

The various dimensionless parameters entering into the equations are

$$R = \frac{\rho U_\infty L}{\mu_v}, \quad E = \frac{U_\infty^2}{C_p(T_w-T_\infty)}, \quad Pr = \frac{\mu_v C_p}{K_c}$$

$$\bar{N}_1 = \frac{k_v}{\mu_v}, \quad \bar{N}_2 = \frac{j}{L^2}, \quad \bar{N}_3 = \frac{\gamma_v}{\mu_v L^2} \quad \text{and} \quad \alpha = \frac{\alpha^* U_\infty}{\mu_v L C_p}$$

where μ_v is the viscosity coefficient, k_v is the vortex viscosity coefficient, γ_v is the spin gradient viscosity coefficient and j is the microinertia density. α^* and K_c are the coefficients of heat conduction. C_p is the specific heat of the fluid at constant pressure and ρ is the mass density. R is the Reynolds number, E is the Eckert number and Pr is the Prandtl number. The quantities $\bar{N}_1$, $\bar{N}_2$, $\bar{N}_3$ and α are the micropolar fluid parameters characterising vortex viscosity, microinertia, spin gradient viscosity and micropolar heat conduction respectively.

We now proceed to carry out the usual boundary-layer approach, as propounded in [5], by fixing the following orders of magnitudes:

$$\begin{aligned} &\bar{N}_1 = N_1, \ \bar{N}_2 = \varepsilon^2 N_2, \ \bar{N}_3 = \varepsilon^2 N_3, \ u = u_0 \\ &v = \varepsilon v_0, \ p = p_0, \ g = \frac{1}{\varepsilon} g_0, \ \theta = \theta_0, \ y = \varepsilon Y \end{aligned} \tag{7}$$

where $\varepsilon = 1/(R)^{1/2}$. This essentially means that $\bar{N}_1$, u, p and θ are of order unity.

In addition, following the principle of least degeneracy as suggested by Van Dyke [7], we further assume that α is also of order unity.

Substituting (7) in (1)–(6) and collecting the coefficients of order unity after taking the Prandtl limit of $\varepsilon \to 0$, x and Y as fixed, the equations for the thermal boundary-layer flow become

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial Y} = 0, \tag{8}$$

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial Y} = -\frac{\partial p_0}{\partial x} + (1+N_1)\frac{\partial^2 u_0}{\partial Y^2} + N_1 \frac{\partial g_0}{\partial Y}, \tag{9}$$

$$\frac{\partial p_0}{\partial Y} = 0, \tag{10}$$

$$N_2\left(u_0 \frac{\partial g_0}{\partial x} + v_0 \frac{\partial g_0}{\partial Y}\right) = N_3 \frac{\partial^2 g_0}{\partial Y^2} - N_1\left(\frac{\partial u_0}{\partial Y} + 2g_0\right), \tag{11}$$

$$\begin{aligned} u_0 \frac{\partial \theta_0}{\partial x} + v_0 \frac{\partial \theta_0}{\partial Y} = &\left(1+\frac{N_1}{2}\right)E\left(\frac{\partial u_0}{\partial Y}\right)^2 + 2N_1 E\left(g_0 + \frac{1}{2}\frac{\partial u_0}{\partial Y}\right)^2 \\ &+ N_3 E\left(\frac{\partial g_0}{\partial Y}\right)^2 + \alpha\left(\frac{\partial \theta_0}{\partial x}\frac{\partial g_0}{\partial Y} - \frac{\partial \theta_0}{\partial Y}\frac{\partial g_0}{\partial x}\right) + \frac{1}{Pr}\frac{\partial^2 \theta_0}{\partial Y^2}. \end{aligned} \tag{12}$$

The terms containing E in the energy equation (12) arise due to frictional heating which is often neglected for incompressible flow.

In view of (10) and Bernoulli's equation for outer flow, we have,

$$-\frac{\partial p_0}{\partial x} = U_0 \frac{dU_0}{dx}$$

where $U_0 = U_0(x)$ is the dimensionless inviscid flow velocity on the surface of the cylinder. The equation (9) therefore becomes

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial Y} = U_0 \frac{dU_0}{dx} + (1+N_1)\frac{\partial^2 u_0}{\partial Y^2} + N_1 \frac{\partial g_0}{\partial Y}. \tag{13}$$

We thus have the four equations, viz. (8), (11), (12) and (13) for the four unknowns u_0, v_0, g_0 and θ_0 in the boundary layer. The inner and outer boundary conditions are

$$u_0 = v_0 = g_0 = 0, \quad \theta_0 = 1 \ \text{on} \ Y = 0 \tag{14a}$$

$$u_0 \to U_0, \quad g_0 \to 0, \quad \theta_0 \to 0 \ \text{as} \ Y \to \infty. \tag{14b}$$

3. METHOD OF SOLUTION

The inviscid flow velocity on the surface of the cylinder is given as $U_0 = 2\sin x$ which is approximated by the following polynomial

$$U_0(x) = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 \tag{15a}$$

where a_1, a_3, a_5 and a_7 depend only on the shape of the body and are considered to be known. In the case of a circular cylinder

$$a_1 = 2, \quad a_3 = -\frac{2}{\lfloor 3}, \quad a_5 = \frac{2}{\lfloor 5} \quad \text{and} \quad a_7 = -\frac{2}{\lfloor 7}.$$

The polynomial (15a) can be regarded as a reasonably good approximation for the function $2\sin x$. The boundary-layer equations break down at the point of separation and so our analysis is valid before the point of separation is reached.

In view of the relation (15a), we write

$$u_0(x,\eta) = \sum_{n=1}^{4} a_{2n-1} x^{2n-1} f'_{2n-1}(\eta), \tag{15b}$$

$$v_0(x,\eta) = -\frac{1}{(a_1)^{1/2}} \sum_{n=1}^{4} (2n-1) a_{2n-1} x^{2n-2} f_{2n-1}(\eta), \tag{15c}$$

$$g_0(x,\eta) = (a_1)^{1/2} \sum_{n=1}^{4} a_{2n-1} x^{2n-1} g_{2n-1}(\eta), \tag{15d}$$

$$\theta_0(x,\eta) = \sum_{n=1}^{4} a_{2n-1} x^{2n-2} \theta_{2n-1}(\eta), \tag{15e}$$

where $\eta = Y(a_1)^{1/2}$.

The functions $f_5, f_7, g_5, g_7, \theta_3, \theta_5$ and θ_7 occurring in equations (15) are further written as:

$$f_5(\eta) = h_5(\eta) + \frac{a_3^2}{a_1 a_5} k_5(\eta), \tag{16a}$$

$$f_7(\eta) = h_7(\eta) + \frac{a_3 a_5}{a_1 a_7} k_7(\eta) + \frac{a_3^3}{a_1^2 a_7} j_7(\eta), \tag{16b}$$

$$g_5(\eta) = l_5(\eta) + \frac{a_3^2}{a_1 a_5} m_5(\eta), \tag{16c}$$

$$g_7(\eta) = l_7(\eta) + \frac{a_3 a_5}{a_1 a_7} m_7(\eta) + \frac{a_3^3}{a_1^2 a_7} n_7(\eta), \tag{16d}$$

$$\theta_3(\eta) = A_3(\eta) + \frac{a_1^2}{a_3} B_3(\eta), \tag{16e}$$

$$\theta_5(\eta) = A_5(\eta) + \frac{a_3^2}{a_1 a_5} B_5(\eta) + \frac{a_1 a_3}{a_5} C_5(\eta), \tag{16f}$$

$$\theta_7(\eta) = A_7(\eta) + \frac{a_3 a_5}{a_1 a_7} B_7(\eta) + \frac{a_3^3}{a_1^2 a_7} C_7(\eta) + \frac{a_1 a_5}{a_7} D_7(\eta) + \frac{a_3^2}{a_7} E_7(\eta). \tag{16g}$$

Substituting (15a–e) in (11)–(13) and using (16a–g), we get the following set of ordinary differential equations on equating the coefficients of like powers of x:

$$(1+N_1) f_1''' + f_1 f_1'' + 1 - f_1'^2 + N_1 g_1' = 0, \tag{17a}$$

$$N_2(f_1' g_1 - f_1 g_1') = N_3 g_1'' - \frac{N_1}{a_1}(f_1'' + 2g_1), \tag{17b}$$

$$(1+N_1) f_3''' + f_1 f_3'' - 4 f_1' f_3' + 3 f_1'' f_3 + 4 + N_1 g_3' = 0, \tag{18a}$$

$$N_2[3(f_1' g_3 - f_3 g_1') + (f_3' g_1 - f_1 g_3')] = N_3 g_3'' - \frac{N_1}{a_1}(f_3'' + 2g_3), \tag{18b}$$

$$(1+N_1) h_5''' + f_1 h_5'' - 6 f_1' h_5' + 5 h_5 f_1'' + 6 + N_1 l_5' = 0, \tag{19a}$$

$$N_2[5(f_1' l_5 - h_5 g_1') + (h_5' g_1 - f_1 l_5')] = N_3 l_5'' - \frac{N_1}{a_1}(h_5'' + 2l_5), \tag{19b}$$

$$(1+N_1) k_5''' + f_1 k_5'' - 6 f_1' k_5' + 5 f_1'' k_5 - 3(f_3'^2 - f_3 f_3'' - 1) + N_1 m_5' = 0, \tag{20a}$$

$$N_2[5(f_1'm_5-k_5g_1')+3(f_3'g_3-f_3g_3')+k_5'g_1-f_1m_5'] = N_3m_5''-\frac{N_1}{a_1}(k_5''+2m_5), \tag{20b}$$

$$(1+N_1)h_7'''+f_1h_7''-8f_1'h_7'+7f_1''h_7+8+N_1l_7' = 0, \tag{21a}$$

$$N_2[7(f_1'l_7-h_7g_1')+(h_7'g_1-f_1l_7')] = N_3l_7''-\frac{N_1}{a_1}(h_7''+2l_7), \tag{21b}$$

$$(1+N_1)k_7'''+f_1k_7''-8f_1'k_7'+7f_1''k_7-8f_3'h_5'+5f_3''h_5+3f_3h_5''+8+N_1m_7' = 0, \tag{22a}$$

$$N_2[7(f_1'm_7-k_7g_1')+5(f_3'l_5-h_5g_3')+3(h_5'g_3-l_5'f_3)+k_7'g_1-f_1m_7'] = N_3m_7''-\frac{N_1}{a_1}(k_7''+2m_7), \tag{22b}$$

$$(1+N_1)j_7'''+f_1j_7''-8f_1'j_7'+7f_1''j_7-8f_3'k_5'+5f_3''k_5+3f_3k_5''+N_1n_7' = 0, \tag{23a}$$

$$N_2[7(f_1'n_7-j_7g_1')+5(f_3'm_5-g_3'k_5)+3(k_5'g_3-f_3m_5')+j_7'g_1-f_1n_7'] = N_3n_7''-\frac{N_1}{a_1}(j_7''+2n_7), \tag{23b}$$

$$\frac{1}{Pr}\theta_1''+f_1\theta_1'-\alpha a_1\theta_1'g_1 = 0, \tag{24}$$

$$\frac{1}{Pr}A_3''+f_1A_3'-2f_1'A_3+3f_3\theta_1'-\alpha a_1(g_1A_3'-2g_1'A_3+3g_3\theta_1') = 0, \tag{25a}$$

$$\frac{1}{Pr}B_3''+f_1B_3'-2f_1'B_3-\alpha a_1(g_1B_3'-2g_1'B_3)+(1+\tfrac{1}{2}N_1)Ef_1''^2+2N_1E(g_1+\tfrac{1}{2}f_1'')^2+N_3Ea_1g_1'^2 = 0, \tag{25b}$$

$$\frac{1}{Pr}A_5''+f_1A_5'-4f_1'A_5+5h_5\theta_1'-\alpha a_1(g_1A_5'-4g_1'A_5+5l_5\theta_1') = 0, \tag{26a}$$

$$\frac{1}{Pr}B_5''+f_1B_5'-4f_1'B_5+3f_3A_3'-2f_3'A_3+5k_5\theta_1'-\alpha a_1(g_1B_5'-4g_1'B_5+3g_3A_3'-2g_3'A_3+5m_5\theta_1') = 0, \tag{26b}$$

$$\frac{1}{Pr}C_5''+f_1C_5'-4f_1'C_5+3f_3B_3'-2f_3'B_3-\alpha a_1(g_1C_5'-4g_1'C_5+3g_3B_3'-2g_3'B_3)$$
$$+2E(1+\tfrac{1}{2}N_1)f_1''f_3''+4N_1E(g_1+\tfrac{1}{2}f_1'')(g_3+\tfrac{1}{2}f_3'')+2EN_3a_1g_1'g_3' = 0, \tag{26c}$$

$$\frac{1}{Pr}A_7''+f_1A_7'-6f_1'A_7+7h_7\theta_1'-\alpha a_1(g_1A_7'-6g_1'A_7+7l_7\theta_1') = 0, \tag{27a}$$

$$\frac{1}{Pr}B_7''+f_1B_7'-6f_1'B_7+3f_3A_5'-4f_3'A_5+5h_5A_3'-2h_5'A_3+7k_7\theta_1'$$
$$-\alpha a_1(g_1B_7'-6g_1'B_7+3g_3A_5'-4g_3'A_5+7m_7\theta_1'+5l_5A_3'-2l_5'A_3) = 0, \tag{27b}$$

$$\frac{1}{Pr}C_7''+f_1C_7'-6f_1'C_7+3f_3B_5'-4f_3'B_5+5k_5A_3'-2k_5'A_3$$
$$+7j_7\theta_1'-\alpha a_1(g_1C_7'-6g_1'C_7+3g_3B_5'-4g_3'B_5+5m_5A_3'-2m_5'A_3+7n_7\theta_1') = 0, \tag{27c}$$

$$\frac{1}{Pr}D_7''+f_1D_7'-6f_1'D_7+5h_5B_3'-2h_5'B_3-\alpha a_1(g_1D_7'-6g_1'D_7+5l_5B_3'-2l_5'B_3)$$
$$+2E(1+\tfrac{1}{2}N_1)f_1''h_5''+4N_1E(g_1+\tfrac{1}{2}f_1'')(l_5+\tfrac{1}{2}h_5'')+2N_3Ea_1g_1'l_5' = 0, \tag{27d}$$

$$\frac{1}{Pr}E_7''+f_1E_7'-6f_1'E_7+3f_3C_5'-4f_3'C_5+5k_5B_3'-2k_5'B_3$$
$$-\alpha a_1(g_1E_7'-6g_1'E_7+3g_3C_5'-4g_3'C_5+5m_5B_3'-2m_5'B_3)+E(1+\tfrac{1}{2}N_1)(2f_1''k_5''+f_3''^2)$$
$$+2N_1E[2(g_1+\tfrac{1}{2}f_1'')(m_5+\tfrac{1}{2}k_5'')+(g_3+\tfrac{1}{2}f_3'')^2]+N_3Ea_1(2g_1'm_5'+g_3'^2) = 0. \tag{27e}$$

The primes in the above equations denote differentiation with respect to η.

In view of the boundary conditions (14), we obtain the following boundary conditions for the set of equations (17)–(27):

$$f_1(0) = f_1'(0) = g_1(0) = 0,\quad f_1'(\infty) = 1,\quad g_1(\infty) = 0, \tag{28}$$

$$f_3(0) = f_3'(0) = g_3(0) = 0,\quad f_3'(\infty) = 1,\quad g_3(\infty) = 0, \tag{29}$$

$$h_5(0) = h_5'(0) = l_5(0) = 0,\quad h_5'(\infty) = 1,\quad l_5(\infty) = 0, \tag{30}$$

$$k_5(0) = k_5'(0) = m_5(0) = 0,\quad k_5'(\infty) = m_5(\infty) = 0, \tag{31}$$

$$h_7(0) = h_7'(0) = l_7(0) = 0,\quad h_7'(\infty) = 1,\quad l_7(\infty) = 0, \tag{32}$$

$$k_7(0) = k_7'(0) = m_7(0) = k_7'(\infty) = m_7(\infty) = 0, \tag{33}$$

$$j_7(0) = j_7'(0) = n_7(0) = j_7'(\infty) = n_7(\infty) = 0, \tag{34}$$

$$\theta_1(0) = \frac{1}{a_1}, \quad A_3(0) = B_3(0) = A_5(0) = B_5(0) = C_5(0) = A_7(0) = B_7(0) = C_7(0) = D_7(0) = E_7(0) = 0, \tag{35a}$$

$$\theta_1(\infty) = A_3(\infty) = B_3(\infty) = A_5(\infty) = B_5(\infty) = C_5(\infty) = A_7(\infty) = B_7(\infty) = C_7(\infty) = D_7(\infty) = E_7(\infty) = 0. \tag{35b}$$

It may be noted that the equations for θ_1, A_3, A_5, B_5, A_7, B_7 and C_7 are free from the terms that arise due to frictional heating. The solutions of B_3, C_5, D_7 and E_7 give the effect of friction on temperature.

The equations (17)–(27) are to be solved with the corresponding boundary conditions (28)–(35). Equation (17a) is non-linear and the remaining equations (17b)–(27) are linear equations. In each group of the coupled equations (17)–(23), the first equation is of third order and the latter is of second order. Each of the equations (24)–(27) is a second order linear differential equation. The relations (28)–(35) furnish boundary conditions for each group of coupled equations. For the functions determined by the equations (17)–(23), three boundary conditions are at $\eta = 0$ and two at $\eta = \infty$. For the functions determined by the equations (24)–(27), one boundary condition is prescribed at $\eta = 0$ and another at $\eta = 0$.

We have solved the coupled equations (17)–(27) numerically using Taylor's series method on CDC 3600 computer with the interval size $\Delta\eta = 0.05$. We illustrate the method for the group of equations (17a) and (17b) for which the boundary conditions are given by (28). To satisfy the boundary conditions (28), three of which are at $\eta = 0$ and the remaining two are given at $\eta = \infty$, we compute the solutions of (17a) and (17b) assuming crude values of $f_1''(0)$ and $g_1'(0)$. These solutions will in general not satisfy the boundary conditions $f_1'(\infty) = 1$ and $g_1(\infty) = 0$. Now these arbitrary values for $f_1''(0)$ and $g_1'(0)$ are changed again and again in a systematic manner till the boundary conditions for large η are satisfied. This is the well known "Shooting Method" of solving a two point boundary value problem. The same method is applied to the remaining equations (18)–(27). The temperature distribution θ_0 is finally evaluated from (15e).

We have assumed $Pr = 1$ and the set of values of N_1, N_2, N_3 and α are recorded on the figures.

4. RESULTS AND DISCUSSIONS

The velocity and microrotation fields of the flow problem considered here have been shown in detail in [8]. Here in this work we have plotted the temperature field.

In most of the present work we have neglected the frictional heating terms because, as we shall see towards the end of this article, they are found insignificant at incompressible speeds. The results in all the figures are therefore obtained without considering the frictional heating terms. It is pertinent to note that if we neglect frictional heating terms, there is no direct influence of the micropolar fluid parameters N_1, N_2 and N_3 on the temperature field. The influence of these parameters on the temperature field enters through the velocity fields.

The temperature profiles have been plotted in Fig. 1. These curves have been drawn at four different stations, viz. $\phi = 30°$ (Fig. 1a), 50° (Fig. 1b), 70° (Fig. 1c) and 105° (Fig. 1d) where ϕ is the angle measured in degrees from the front stagnation point. We have considered the five sets of values for N_1, N_2 and N_3: (i) $N_1 = 4.5$, $N_2 = 9.0$, $N_3 = 13.5$; (ii) $N_1 = 13.5$, $N_2 = 9.0$, $N_3 = 13.5$; (iii) $N_1 = 4.5$, $N_2 = 40.5$, $N_3 = 13.5$; (iv) $N_1 = 4.5$, $N_2 = 9.0$, $N_3 = 40.5$; (v) $N_1 = N_2 = N_3 = \alpha = 0$ (Newtonian fluid).

For the first four sets of values of N_1, N_2 and N_3, we also examined the effect of variation of α by considering two values of α, viz. $\alpha = 0$ and $\alpha = 1$.

The sets (i) and (ii) give the effect of variation of N_1 when N_2 and N_3 are kept constant. The temperature at a given η-station increases with the increase of N_1 for both $\alpha = 0$ and $\alpha = 1$. This increase is more pronounced as we move in the down stream direction.

The sets (i) and (iii) give the effect of variation of N_2 when N_1 and N_3 are kept constant. Except at $\phi = 105°$, at all other ϕ stations, there is hardly any appreciable effect of N_2 variation on the temperature profile for $\alpha = 0$. In case of $\alpha = 1$ the effect of the increase of N_2 is to increase the temperature at a given η-station.

The sets (i) and (iv) give the effect of variation of N_3 when N_1 and N_2 are kept constant. There is hardly any change in the temperature for $\alpha = 0$ at $\phi = 30°$, 50° and 70° when N_3 is varied. For $\alpha = 0$ at $\phi = 105°$ and $\alpha = 1$ at all ϕ stations, the temperature increases with the increase of N_3.

Generally [except for the set (ii) at $\phi = 105°$] the temperature at a particular station η is less for $\alpha = 1$ as compared to $\alpha = 0$. This difference of temperature becomes more pronounced with the increase of N_1 as compared to the increase of N_2 or N_3.

All these profiles (Figs. 1a–d) of the micropolar fluid are compared with the corresponding profiles for the Newtonian fluid. ($N_1 = N_2 = N_3 = \alpha = 0$). The Newtonian temperature profiles have been plotted with dotted lines. It is clear that at any given station η, the effect of the material parameters is to increase the temperature as compared to the corresponding flow of a Newtonian fluid.

Now we proceed to examine the heat flux at the wall. The non-dimensional heat-transfer coefficient, called the

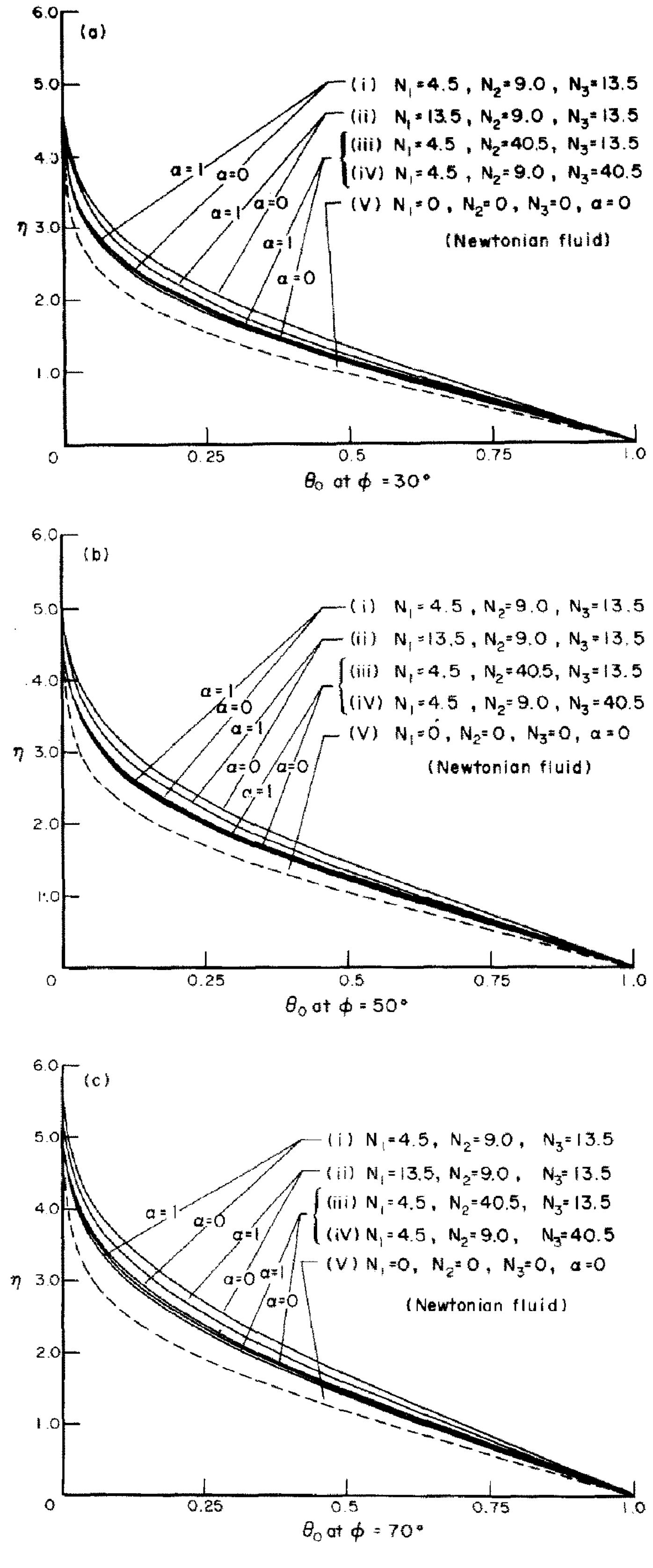
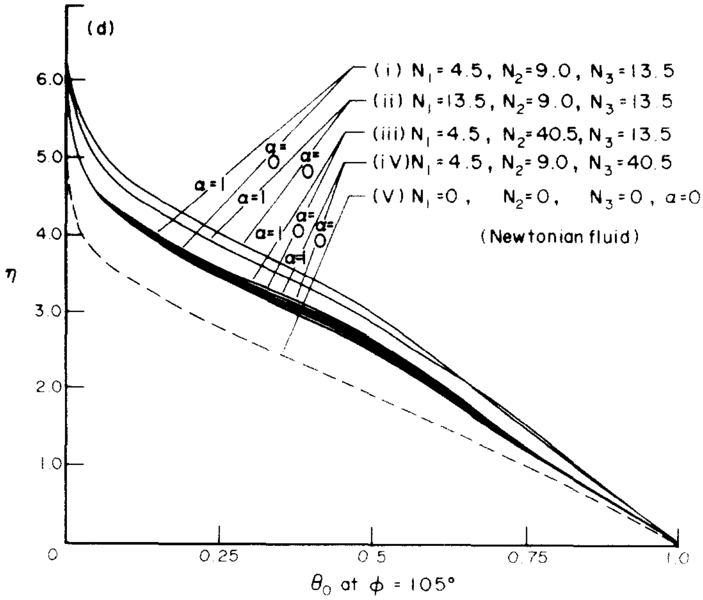


Fig. 1(a). Effect of variation of N_1, N_2, N_3 and α on the temperature profiles at $\phi = 30°$. (b) Effect of variation of N_1, N_2, N_3 and α on the temperature profiles at $\phi = 50°$. (c) Effect of variation of N_1, N_2, N_3 and α on the temperature profiles at $\phi = 70°$. (d) Effect of variation of N_1, N_2, N_3 and α on the temperature profiles at $\phi = 105°$.



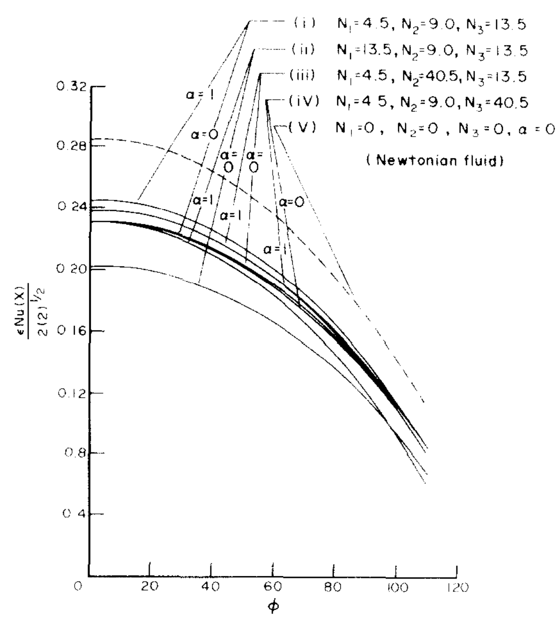


FIG. 2. Effect of variation of N_1, N_2, N_3 and α on the heat-transfer coefficient parameter $\varepsilon Nu(x)/2(2)^{1/2}$.

Nusselt number, is defined as follows:

$$Nu(x) = \frac{qL}{K_c(T_w - T_\infty)} = -\left(\frac{\partial\theta_0}{\partial y}\right)_{y=0} = -\frac{1}{\varepsilon}\left(\frac{\partial\theta_0}{\partial Y}\right)_{Y=0},$$

where q is the heat flux at the wall. Therefore we can write,

$$\frac{\varepsilon Nu(x)}{(a_1)^{1/2}} = -\left(\frac{\partial\theta_0}{\partial\eta}\right)_{\eta=0} = -\sum_{n=1}^{4} a_{2n-1}x^{2n-2}\theta'_{2n-1}(0).$$

The heat-transfer coefficient parameter $\varepsilon Nu(x)/2(2)^{1/2}$ for the same four sets of values (as for temperature profiles) of N_1, N_2 and N_3 have been plotted in Fig. 2. For each of these sets, heat transfer at the wall is again plotted for two different values of α, viz. $\alpha = 0$ and $\alpha = 1$.

The heat-transfer coefficient is found to decrease with the increase of N_1 for $\alpha = 0$ and $\alpha = 1$. With the increase of N_2, the heat-transfer coefficient increases for $\alpha = 0$ and decreases for $\alpha = 1$. When N_3 is increased the heat-transfer coefficient decreases for $\alpha = 1$ and it has hardly any change for $\alpha = 0$.

Generally (except for $N_1 = 13.5, N_2 = 9.0, N_3 = 13.5$ after $\phi = 98°$), the heat-transfer coefficient is more for $\alpha = 1$ as compared to $\alpha = 0$. This difference becomes less pronounced as we move away from the stagnation point. The heat-transfer coefficient parameter for Newtonian fluid is plotted with the dotted lines and it is clear that the effect of micropolar fluid parameters is to decrease the heat-transfer coefficient.

Comparing figures for the heat-transfer coefficient (Fig. 2) and the temperature fields (Figs. 1a–d), we note that the heat-transfer coefficient for Newtonian fluid is more and the temperature is less when compared with that for micropolar fluids. This can be explained as follows. The temperature $\theta_0(x, \eta)$ at any point (x, η) inside the boundary layer at a small distance η from the wall can be approximately written as

$$\theta_0(x, \eta) = \theta_0(x, 0) + \theta_0'(x, 0)\eta = 1 - \frac{\varepsilon}{(2)^{1/2}} Nu(x)\eta.$$

The difference between the temperatures of a Newtonian fluid and a micropolar fluid at the same point inside the boundary layer at a distance η from the wall can be written as

$$[\theta(x, \eta)]_{\text{Newtonian}} - [\theta(x, \eta)]_{\text{micropolar}} = \{[Nu(x)]_{\text{micropolar}} - [Nu(x)]_{\text{Newtonian}}\} \frac{\varepsilon}{(2)^{1/2}} \eta.$$

Table 1. Temperature distribution θ_0 for $N_1 = 4.5$, $N_2 = 9.0$, $N_3 = 13.5$, $\alpha = 0$, $Pr = 1$ and $E = 0.01$ at $\phi = 30°$, 70° and 105° for the two cases—neglecting frictional heating and including frictional heating

	$\phi = 30°$		$\phi = 70°$		$\phi = 105°$	
η	θ_0 (Neglecting frictional heating)	θ_0 (Including frictional heating)	θ_0 (Neglecting frictional heating)	θ_0 (Including frictional heating)	θ_0 (Neglecting frictional heating)	θ_0 (Including frictional heating)
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.5	0.7801	0.7826	0.8261	0.8333	0.9053	0.9119
1.0	0.5668	0.5696	0.6543	0.6640	0.8092	0.8120
1.5	0.3743	0.3765	0.4003	0.5000	0.7118	0.7192
2.0	0.2193	0.2207	0.3427	0.3509	0.6162	0.6321
2.5	0.1114	0.1122	0.2200	0.2259	0.5178	0.5373
3.0	0.0481	0.0484	0.1269	0.1306	0.4007	0.4175
3.5	0.0173	0.0175	0.0639	0.0659	0.2650	0.2759
4.0	0.0052	0.0052	0.0272	0.0281	0.1407	0.1465
4.5	0.0012	0.0013	0.0094	0.0098	0.0579	0.0605
5.0	0.0002	0.0003	0.0026	0.0027	0.0181	0.0192
5.5	0.0000	0.0000	0.0006	0.0006	0.0043	0.0047
6.0			0.0001	0.0001	0.0008	0.0010
6.5			0.0000	0.0000	0.0001	0.0002
7.0					0.0000	0.0001

Table 2. Temperature distribution θ_0 for $N_1 = 4.5$, $N_2 = 9.0$, $N_3 = 13.5$, $\alpha = 1$, $Pr = 1$ and $E = 0.01$ at $\phi = 30°$, 70° and 105° for the two cases—neglecting frictional heating and including frictional heating

	$\phi = 30°$		$\phi = 70°$		$\phi = 105°$	
η	θ_0 (Neglecting frictional heating)	θ_0 (Including frictional heating)	θ_0 (Neglecting frictional heating)	θ_0 (Including frictional heating)	θ_0 (Neglecting frictional heating)	θ_0 (Including frictional heating)
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.5	0.7680	0.7704	0.8170	0.8241	0.9026	0.9092
1.0	0.5475	0.5503	0.6384	0.6480	0.8030	0.8062
1.5	0.3548	0.3570	0.4713	0.4809	0.7013	0.7089
2.0	0.2043	0.2057	0.3244	0.3325	0.5998	0.6154
2.5	0.1023	0.1031	0.2052	0.2110	0.4944	0.5137
3.0	0.0437	0.0441	0.1168	0.1204	0.3742	0.3908
3.5	0.0157	0.0158	0.0582	0.0601	0.2429	0.2538
4.0	0.0046	0.0047	0.0245	0.0254	0.1275	0.1332
4.5	0.0011	0.0011	0.0085	0.0088	0.0521	0.0546
5.0	0.0002	0.0002	0.0023	0.0025	0.0163	0.0173
5.5	0.0000	0.0000	0.0005	0.0005	0.0038	0.0042
6.0			0.0001	0.0001	0.0007	0.0009
6.5			0.0000	0.0000	0.0001	0.0002
7.0					0.0000	0.0001

Since $[Nu(x)]_{\text{Newtonian}}$ is greater than $[Nu(x)]_{\text{micropolar}}$, therefore, the temperature of a micropolar fluid is more than the temperature of a Newtonian fluid in view of the above relation.

Towards the end of the present work, we have also computed the temperature distribution θ_0 and heat-transfer coefficient parameter $\varepsilon Nu(x)/2(2)^{1/2}$ including the frictional heating terms. The magnitude of the Eckert number E is taken as 0.01 and the case we considered is for $N_1 = 4.5$, $N_2 = 9.0$, $N_3 = 13.5$. These values have been recorded in Table 1 (for $\alpha = 0$) and in Table 2 (for $\alpha = 1$) for temperature fields at $\phi = 30°$, $70°$ and $105°$ and in Table 3 (for $\alpha = 0$ and $\alpha = 1$) for heat-transfer coefficient. It is clear from these tables that the inclusion of

Table 3. Heat-transfer coefficient parameter $\varepsilon Nu(x)/2(2)^{1/2}$ for $N_1 = 4.5$, $N_2 = 9.0$, $N_3 = 13.5$, $Pr = 1$ and $E = 0.01$ (for $\alpha = 0$ and $\alpha = 1$) for the two cases—neglecting frictional heating and including frictional heating

	$\alpha = 0$		$\alpha = 1$	
ϕ	$\varepsilon Nu(x)/2(2)^{1/2}$ (Neglecting frictional heating)	$\varepsilon Nu(x)/2(2)^{1/2}$ (Including frictional heating)	$\varepsilon Nu(x)/2(2)^{1/2}$ (Neglecting frictional heating)	$\varepsilon Nu(x)/2(2)^{1/2}$ (Including frictional heating)
0	0.2305	0.2305	0.2445	0.2445
10	0.2293	0.2288	0.2433	0.2428
20	0.2260	0.2241	0.2396	0.2378
30	0.2204	0.2164	0.2336	0.2297
40	0.2124	0.2063	0.2251	0.2190
50	0.2022	0.1940	0.2140	0.2059
60	0.1894	0.1797	0.2006	0.1906
70	0.1740	0.1633	0.1836	0.1730
80	0.1557	0.1440	0.1638	0.1522
90	0.1342	0.1201	0.1404	0.1265
100	0.1088	0.0888	0.1128	0.0931

the frictional heating terms has no appreciable influence and thereby supports the assumption of negligible frictional heating in the present work.

To sum up, we can therefore, state that the effect of variation of N_1 is more pronounced as compared to the variation of either N_2 or N_3. Generally [except for the set (ii) of values of N_1, N_2 and N_3], the temperature is less and the heat-transfer coefficient is more for $\alpha = 1$ as compared to $\alpha = 0$. It is seen that the effect of micropolar fluid parameters is to increase the temperature inside the boundary layer and to decrease the heat-transfer coefficient as compared to Newtonian fluid.

Acknowledgement—Mrs. P. Subhadra Ramachandran is grateful to C.S.I.R. Government of India for the award of a research fellowship which helped in completing this work.

REFERENCES

1. J. W. Hoyt and A. G. Fabula, U.S. Naval Ordnance Test Station Report (1964).
2. W. M. Vogel and A. M. Patterson, Pacific Naval Lab. of the Defence Res. Board of Canada Report, 64-2 (1964).
3. A. C. Eringen, Mechanics of micropolar continua, ONR Report (1965).
4. A. C. Eringen, Theory of thermomicrofluids, *J. Math. Anal. Applic.* **38**, 480–496 (1972).
5. J. Peddieson, Jr. and R. P. McNitt, Boundary layer theory for a micropolar fluid, *Rec. Adv. Engng Sci.* **5**, 405–426 (1970).
6. H. Schlichting, *Boundary Layer Theory*, Chapter 9. McGraw-Hill, New York (1968).
7. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, Chapter 5. Academic Press, London (1964).
8. A. K. Banerjee, S. K. Ojha and M. N. Mathur, Boundary layer flow of a micropolar fluid past a circular cylinder, *Iranian J. Sci. Technol.*, to be published.

COUCHE LIMITE THERMIQUE D'UN FLUIDE MICROPOLAIRE SUR UN CYLINDRE CIRCULAIRE

Résumé—On étudie la couche limite stationnaire sur un cylindre circulaire dont l'axe est normal à un écoulement de fluide micropolaire et incompressible. On obtient la solution de l'équation d'énergie dans la couche limite sous la forme d'un développement en série de la distance curviligne au point d'arrêt amont. La surface du cylindre est maintenue à température constante de même que l'écoulement libre. La distribution de température adimansionnelle et le coefficient de transfert de chaleur sont présentés graphiquement pour plusieurs valeurs des paramètres. On les compare aux résultats correspondants pour les fluides Newtoniens. Pour les fluides micropolaires, la température dans la couche limite est plus grande que pour les fluides Newtoniens alors que le coefficient de transfert est plus faible.

DIE THERMISCHE GRENZSCHICHT EINES MIKROPOLAREN FLUIDS AM KREISZYLINDER

Zusammenfassung—Untersucht wurde die stationäre thermische Grenzschichtströmung hinter einem Kreiszylinder, dessen Achse senkrecht in der freien Zuströmung eines inkompressiblen, mikropolaren Fluids steht. Die Lösung der Energiegleichung, angewandt auf die Grenzschicht, erhält man als Exponentialfunktion der Umströmungslänge entlang der Zylinderoberfläche vom Staupunkt aus. Sowohl die Oberflächentemperatur des Kreiszylinders als auch die Temperatur außerhalb der Grenzschicht wurden als konstant angenommen. Die dimensionslose Temperaturverteilung und der Wärmeübergangskoeffizient wurden grafisch aufgetragen für verschiedene Werte der Materialparameter. Die Ergebnisse werden verglichen mit entsprechenden Werten für Newton'sche Flüssigkeiten. Es zeigt sich, daß bei mikropolaren Substanzen die Grenzschicht-Temperaturgrößen, der Wärmeübergangskoeffizient kleiner ist als bei Newton'schen Flüssigkeiten.

ИССЛЕДОВАНИЕ ПОГРАНИЧНОГО СЛОЯ МИКРОПОЛЯРНОЙ ЖИДКОСТИ НА ПОВЕРХНОСТИ КРУГЛОГО ЦИЛИНДРА

Аннотация — В работе исследуется стационарный теплообмен в пограничном слое на круглом цилиндре, ось которого перпендикулярна набегающему свободному потоку несжимаемой микрополярной жидкости. Решение уравнения энергии получено в виде степенных рядов по координате, измеренной по поверхности от лобовой критической точки цилиндра. Температуры круглого цилиндра и набегаюшего потока считаются постоянными. Графически представлено распределение безразмерной температуры и коэффициента теплообмена для различных значений параметров среды. Было проведено сопоставление с соответствующими данными для ньютоновских жидкостей. У микрополярных жидкостей, по сравнению с ньютоновскими, температура внутри пограничного слоя больше, а коэффициент теплообмена меньше.